

Phase transition for the dilute clock model

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Abstract

We prove that phase transition occurs in the dilute ferromagnetic nearest-neighbour q -state clock model in \mathbb{Z}^d , for every $q \geq 2$ and $d \geq 2$. This follows from the fact that the Edwards-Sokal random-cluster representation of the clock model stochastically dominates a supercritical Bernoulli bond percolation probability, a technique that has been applied to show phase transition for the low-temperature Potts model. The domination involves a combinatorial lemma which is one of the main points of this article.

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1 Introduction

The q -state clock model assigns a random spin to each site of \mathbb{Z}^d . The spins take values in a discrete set S of equidistant angles or hours, hence the name. Let $\sigma = (\sigma_x, x \in \mathbb{Z}^d)$ be a spin configuration, σ_x the angle of the spin at $x \in \mathbb{Z}^d$. Let $\mathcal{E}(\mathbb{Z}^d) := \{\langle xy \rangle : \|x - y\| = 1\}$ be the set of edges connecting nearest neighbour sites, $\|\cdot\|$ the Euclidean norm. We study the dilute clock model associated to a disorder, namely a collection

$$J = (J_{\langle xy \rangle} : \langle xy \rangle \in \mathcal{E}(\mathbb{Z}^d)) \quad (1)$$

of independent identically distributed Bernoulli random variables with parameter p . A disorder realization J and a finite set $\Lambda \subset \mathbb{Z}^d$ determine the Hamiltonian on spin configurations:

$$H_{\Lambda, J}(\sigma) := \sum_{\substack{\langle xy \rangle \in \mathcal{E}(\mathbb{Z}^d) \\ \{x, y\} \cap \Lambda \neq \emptyset}} J_{\langle xy \rangle} (1 - \cos(\sigma_x - \sigma_y)). \quad (2)$$

When $q = 2$, we recover the Ising model; as $q \rightarrow \infty$, the clock model approximates the XY model, which has a continuum of spin angles.

Given a set $\Lambda \subset \mathbb{Z}^d$ and configurations $\sigma, \eta \in S^{\mathbb{Z}^d}$, we write

$$\sigma \stackrel{\Lambda}{=} \eta \quad \text{if} \quad \sigma_x = \eta_x \quad \forall x \in \Lambda. \quad (3)$$

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The configuration η plays the role of a boundary condition. The specification $\mu_{\Lambda, J}^\eta$ associated to a finite set Λ , a disorder J , and a boundary condition η is the probability

$$\mu_{\Lambda, J}^\eta(\sigma) := \frac{1}{Z_{\Lambda, J}^\eta} e^{-\beta H_{\Lambda, J}(\sigma)} \mathbf{1}[\sigma|_{\Lambda^c} = \eta], \quad (4)$$

where $\beta > 0$ is a parameter proportional to the inverse temperature, $Z_{\Lambda, J}^\eta$ is the normalizing constant and $\Lambda^c = \mathbb{Z}^d \setminus \Lambda$. A Gibbs measure associated to the disorder J is a probability μ_J that satisfies the DLR condition:

$$\mu_J f = \int_{S^{\mathbb{Z}^d}} \mu_J(d\eta) \mu_{\Lambda, J}^\eta f \quad (5)$$

for every finite subset $\Lambda \subset \mathbb{Z}^d$ and every local function $f : S^{\mathbb{Z}^d} \rightarrow \mathbb{R}$. Here, μf denotes the expectation of f with respect to μ . The underlying σ -algebra where the Gibbs measures and the specifications are defined is the one generated by projections over finite subsets of \mathbb{Z}^d . We call \mathcal{G}_J the set of Gibbs measures associates to J . Since S is finite, \mathcal{G}_J is not empty. In case $|\mathcal{G}_J| > 1$, we say that phase co-existence occurs.

The homogeneous version of the model is obtained by taking $p = 1$ or, equivalently, $J_{\langle xy \rangle} \equiv 1$ for every $\langle xy \rangle$. In this case, non-uniqueness methods such as the Pirogov-Sinai theory [PS75] or reflection positivity as in Frölich, Israel, Lieb and Simon [FILS78], see also Biskup [Bis09], prove that, for sufficiently low temperature, there exist at least q different Gibbs measures. On the other hand, when the temperature is large enough, techniques similar to those developed by Dobrushin [Dob68] or by van den Berg and Maes [vdBM94] show that there exists only one Gibbs measure. Phase transition occurs when a system undergoes a change in its phase diagram depending on the value of a parameter; these results hence establish occurrence of phase transition for the homogeneous clock model.

Both Pirogov-Sinai theory and reflection positivity depend on the graph determined by the interacting edges in the Hamiltonian (2) being symmetric, an assumption that breaks down for the properly dilute model $p < 1$. Instead, our main tool is the Fortuin-Kasteleyn random-cluster representation [KF69], originally introduced for the Ising and the Ashkin-Teller-Potts models, and then generalized to arbitrary models by Edwards and Sokal [ES88]; here, we build the clock model random-cluster representation in detail. The core idea of this approach is to relate non-uniqueness of Gibbs measure in the statistical-mechanical model to the existence of an infinite cluster in the random-cluster model: a percolation problem. It was first applied by Aizenman, Chayes, Chayes and Newman to study the phase diagram of the dilute Ising and Potts models in [ACCN87]; we presently adapt their ideas to our context.

Precisely, we derive a lower bound for the critical temperature: for every dimension d and every number q of spins, we take p sufficiently large to guarantee that the disorder almost surely contains an infinite bond-percolation cluster, and then determine a value $\beta_0 = \beta_0(q, d, p) > 0$ such that there is more than one Gibbs measure at inverse temperatures $\beta > \beta_0$, for almost all disorders J . A crucial step in the proof consists of dominating the random-cluster probability associated to the clock model by a supercritical Bernoulli product probability on the bonds. While for the Potts model this domination is immediate, the clock model requires a combinatorial argument, given in Lemma 2.4.

The following is our main result.

Theorem 1.1. *Let $p > p_c$, where p_c is the Bernoulli bond percolation critical probability in \mathbb{Z}^d . Let the disorder J be distributed as a product P_p of i.i.d. Bernoulli random variables with parameter p . Then there exists $\beta_0 > 0$ such that, for $\beta > \beta_0$, the q -state dilute clock model associated to the random specifications $\mu_{\Lambda, J}^\eta$ defined in (4) exhibits phase co-existence for P_p -almost every realization of J . More precisely, $\beta > \beta_0$ implies $P_p(J : |\mathcal{G}_J| \geq q) = 1$.*

The value β_0 in the later theorem depends on d , q and p . We show in the Appendix that, for fixed d and p , $\beta_0(q, d, p) \sim q^2 \log q$ as $q \rightarrow \infty$, the same asymptotics provided by Pigorov-Sinai theory and reflection positivity in the 2-dimensional homogeneous case. In particular, $\lim_{q \rightarrow \infty} \beta_0(q, d, p) = \infty$, implying that our approach is not suitable to study the XY model; see van Enter, K\"{u}lske and Opoku [vEKO11] for results concerning the approximation of the XY model via the clock model. On the other hand, for $d \geq 3$ and $p = 1$, reflection positivity computes a threshold β_0 independent of q , see Maes and Shlosman [MS11] for a discussion.

The ideas presented in this article can be further developed in two directions, which are explored by Soprano-Loto in collaboration with Roberto Fern\'andez in a separate article [FSL]. The first one is a generalization of the current work to the so called Abelian spin models; see Dub\'edat [Dub11] for a precise definition. The second direction of research seeks to obtain a uniqueness criterion, also via random-cluster representation, at a higher level of generality.

Organization of the article. We introduce the random-cluster model and state the results leading to the proof of Theorem 1.1 in Section 2. Section 3 contains the proofs and the Appendix collects some auxiliary computations.

2 Clock model and random-cluster in a finite graph

We define the clock model and its random-cluster representation for a fixed non-oriented finite graph (V, \mathcal{E}) without loops or multiple edges, and not necessarily connected. We fix a non-empty subset $U \subset V$ playing the role of *boundary*. For simplicity, we suppose there are no edges connecting vertices in U : $\{\langle xy \rangle \in \mathcal{E} : \{x, y\} \subset U\} = \emptyset$.

In the case of the dilute clock in a finite set $\Lambda \subset \mathbb{Z}^d$, the boundary is given by $\partial\Lambda := \{y \in \mathbb{Z}^d \setminus \Lambda, \exists x \in \Lambda : \|x - y\|^2 = 1\}$, and the vertex and edge sets are

$$\Lambda \cup \partial\Lambda \quad \text{and} \quad \{\langle xy \rangle, \{x, y\} \not\subset \Lambda^c, \|x - y\| = 1, J_{\langle xy \rangle} = 1\}. \quad (6)$$

The clock model. Let S be the set of angles defined by

$$S := \left\{ \frac{2\pi i}{q} : i = 0, \dots, q-1 \right\}. \quad (7)$$

Elements of S are called *spins* and denoted a , b and c , while *spin* or *vertex-configurations* in S^V are denoted by σ and η .

The clock Hamiltonian $H = H(V, \mathcal{E})$ is the function $H : S^V \rightarrow \mathbb{R}$ defined by

$$H(\sigma) := \sum_{\langle xy \rangle \in \mathcal{E}} (1 - \cos(\sigma_x - \sigma_y)). \quad (8)$$

We write $\sigma \stackrel{U}{=} a$ when $\sigma_x = a$ for all $x \in U$. The clock probability $\mu = \mu(V, U, \mathcal{E}, \beta)$ with 0-boundary condition is defined as

$$\mu(\sigma) := \frac{1}{Z} e^{-\beta H(\sigma)} \mathbf{1}[\sigma \stackrel{U}{=} 0], \quad (9)$$

where β is a strictly positive parameter and $Z = Z(V, U, \mathcal{E}, \beta)$ is the normalizing constant.

The random-cluster measure. Define a weight function $W : S \rightarrow (0, 1]$ by

$$W(a) := e^{-\beta(1 - \cos a)} \quad (10)$$

and let $\mathcal{I} := \{W(a), a \in S\}$ be its image. This set has cardinality $|\mathcal{I}| = k + 1$, where $k = q/2$ for even q and $k = (q - 1)/2$ for odd q . Write $\mathcal{I} = \{t_0, t_1, \dots, t_k\}$ with $0 < t_0 < t_1 < \dots < t_k = W(0) = e^{-\beta(1-\cos 0)} = 1$, and denote

$$r_0 := t_0, \quad r_i := t_i - t_{i-1}, \quad 1 \leq i \leq k. \quad (11)$$

By construction $0 \leq r_i \leq 1$ for all $0 \leq i \leq k$, and $\sum_i r_i = 1$.

Let θ be the probability on \mathcal{I} given by

$$\theta(t_i) := r_i, \quad 0 \leq i \leq k, \quad (12)$$

and let $\hat{\phi} = \hat{\phi}(\mathcal{E}, \beta)$ be the product measure on the set of edge-configurations $\omega \in \mathcal{I}^{\mathcal{E}}$ with marginals θ :

$$\hat{\phi}(\omega) := \prod_{\langle xy \rangle \in \mathcal{E}} \theta(\omega_{\langle xy \rangle}). \quad (13)$$

We say that an edge-configuration $\omega \in \mathcal{I}^{\mathcal{E}}$ and a vertex-configuration $\sigma \in S^V$ are *compatible*, and write $\omega \preceq \sigma$, if the value of ω on any edge is dominated by the weight of the gradient of σ over that edge:

$$\omega \preceq \sigma \quad \Leftrightarrow \quad \omega_{\langle xy \rangle} \leq W(\sigma_x - \sigma_y) \text{ for every } \langle xy \rangle \in \mathcal{E}. \quad (14)$$

Notice that if $\omega \preceq \sigma$ and $\omega_{\langle xy \rangle} = 1$, then $\sigma_x = \sigma_y$; on the other hand, $\omega_{\langle xy \rangle} = 0$ imposes no restriction on the values of σ_x and σ_y .

We define the *random-cluster probability* $\phi = \phi(V, U, \mathcal{E}, \beta)$ on $\mathcal{I}^{\mathcal{E}}$ as the measure obtained from $\hat{\phi}$ by assigning to each edge-configuration ω a weight proportional to the number of vertex-configurations σ that are compatible with ω and satisfy the boundary condition, using $\hat{\phi}$ as reference measure:

$$\phi(\omega) := \frac{1}{Z} |\{\sigma : \sigma \succeq \omega, \sigma \stackrel{U}{=} 0\}| \hat{\phi}(\omega). \quad (15)$$

Here Z is the same normalizing constant appearing in (9).

The Edwards-Sokal coupling. Let $\hat{\mu} = \hat{\mu}(V, U)$ be the uniform probability on the set of vertex configurations S^V that are identically 0 at sites in U :

$$\hat{\mu}(\sigma) := \frac{1}{q^{|V \setminus U|}} \mathbf{1}[\sigma \stackrel{U}{=} 0]. \quad (16)$$

We define a joint edge-vertex probability $Q = Q(V, U, \mathcal{E}, \beta)$ on $\mathcal{I}^{\mathcal{E}} \times S^V$ by

$$Q(\omega, \sigma) := \frac{1}{Z'} \mathbf{1}[\omega \preceq \sigma] \hat{\phi}(\omega) \hat{\mu}(\sigma), \quad (17)$$

where $Z' := Z/q^{|V \setminus U|}$ with Z as in (9). That is, Q is the product probability $\hat{\phi} \times \hat{\mu}$ conditioned to the compatibility event $\{(\omega, \sigma) : \omega \preceq \sigma\} \subset \mathcal{I}^{\mathcal{E}} \times S^V$.

Theorem 2.1. *Edwards-Sokal [ES88].*

The measures ϕ and μ are respectively the first and second marginals of Q .

We prove this theorem in Section 3. The measure Q can be seen as a coupling between the clock measure μ and the random-cluster measure ϕ . As a corollary, it follows that the conditional distribution under Q of σ given ω is uniform on the set of configurations compatible with ω and such that $\sigma \stackrel{U}{=} 0$:

$$Q(\sigma | \omega) = \frac{Q(\omega, \sigma)}{\sum_{\sigma'} Q(\omega, \sigma')} = \frac{\hat{\mu}(\sigma) \mathbf{1}[\omega \preceq \sigma]}{\hat{\mu}(\sigma' : \omega \preceq \sigma')}. \quad (18)$$

This implies that a random vertex-configuration distributed according to μ may be sampled by first choosing an edge-configuration ω with law ϕ , and then sampling a vertex-configuration uniformly among those that are compatible with ω and satisfy the boundary restriction. That is,

$$\mu(\sigma) = \sum_{\omega \in \mathcal{I}^{\mathcal{E}}} \frac{\hat{\mu}(\sigma) \mathbf{1}[\omega \preceq \sigma]}{\hat{\mu}(\sigma' : \omega \preceq \sigma')} \phi(\omega). \quad (19)$$

Given $x, y \in V$ and $\omega \in \mathcal{I}^{\mathcal{E}}$, we denote $x \stackrel{\omega}{\longleftrightarrow} y$ if there is a path of vertices $x_1, \dots, x_n \in V$ with $x_1 = x$, $x_n = y$, $\langle x_i x_{i+1} \rangle \in \mathcal{E}$ and $\omega_{\langle x_i x_{i+1} \rangle} = 1$ for $1 \leq i \leq n-1$. We say that x is *connected* to U by an ω -open path, and write $x \stackrel{\omega}{\longleftrightarrow} U$, when $x \stackrel{\omega}{\longleftrightarrow} y$ for some $y \in U$; let $x \stackrel{\omega}{\nleftrightarrow} U$ denote the complementary event. The μ -marginal of the spin at x can be related to the connection probabilities between x and the boundary, under ϕ and Q :

$$\mu(\sigma : \sigma_x = a) = \phi(\omega : x \stackrel{\omega}{\longleftrightarrow} U) \mathbf{1}[a = 0] + Q((\omega, \sigma) : \sigma_x = a, x \stackrel{\omega}{\nleftrightarrow} U). \quad (20)$$

Identity (20) follows immediately from the coupling of Theorem 2.1 and the inclusion $\{(\omega, \sigma) : x \stackrel{\omega}{\longleftrightarrow} U\} \subset \{(\omega, \sigma) : \sigma_x = 0\}$.

The coupling of Theorem 2.1 also implies that the μ -probability of seeing a 0 at any site x is larger than the probability of seeing any other spin plus the ϕ -probability that x be connected to the boundary. This is the content of the next result; its proof depends crucially on the combinatorial Lemma 2.4 stated later.

Proposition 2.2. *Positive correlations.*

For any vertex $x \in V$ and any spin $a \neq 0$,

$$\mu(\sigma : \sigma_x = 0) \geq \mu(\sigma : \sigma_x = a) + \phi(\omega : x \stackrel{\omega}{\longleftrightarrow} U). \quad (21)$$

Stochastic domination. Given $I \subset \mathbb{R}$, consider the partial order on $I^{\mathcal{E}}$ defined by $\omega \leq \omega'$ if and only if $\omega_{\langle xy \rangle} \leq \omega'_{\langle xy \rangle}$ for every $\langle xy \rangle \in \mathcal{E}$. A function $f : I^{\mathcal{E}} \rightarrow \mathbb{R}$ is said to be increasing if $f(\omega) \leq f(\omega')$ whenever $\omega \leq \omega'$, while an event $E \subset I^{\mathcal{E}}$ is said to be increasing when its indicator function $f(\omega) = \mathbf{1}[\omega \in E]$ is. Given two probabilities P and P' on $I^{\mathcal{E}}$, we say that P is stochastically dominated by P' , and write $P \leq_{st} P'$, if and only if $Pf \leq P'f$ for every increasing $f : I^{\mathcal{E}} \rightarrow \mathbb{R}$. This is equivalent to $P(E) \leq P'(E)$ for any increasing event E .

Given $\rho \in [0, 1]$, let B_ρ be the Bernoulli product measure on $\{0, 1\}^{\mathcal{E}}$ with parameter ρ . In order to stochastically compare ϕ and B_ρ we consider them defined on the common space $I^{\mathcal{E}}$, where $I = \{0\} \cup \mathcal{I}$.

Theorem 2.3. *Stochastic domination.*

For any $\rho \in [0, 1)$ there exists $\beta_0 = \beta_0(\rho) > 0$, independent of the graph (V, \mathcal{E}) and the boundary U , such that, if $\beta \geq \beta_0$, B_ρ is stochastically dominated by ϕ .

The key to the proofs of Proposition 2.2 and Theorem 2.3 is the following combinatorial lemma, proved in Section 3.

Lemma 2.4. *For every $x \in V$, $a \in S$ and $\omega \in \mathcal{I}^\varepsilon$,*

$$|\{\sigma : \sigma \preceq \omega, \sigma \stackrel{U}{=} 0, \sigma_x = a\}| \leq |\{\sigma : \sigma \preceq \omega, \sigma \stackrel{U}{=} 0, \sigma_x = 0\}|. \quad (22)$$

Equivalently,

$$\hat{\mu}(\sigma : \sigma_x = a, \sigma \preceq \omega) \leq \hat{\mu}(\sigma : \sigma_x = 0, \sigma \preceq \omega). \quad (23)$$

The lemma in fact holds for any spin set S' and weight function W' provided they satisfy certain symmetry properties: for any pair of elements $a, b \in S'$ it must be possible to define a reflection $R = R_{a,b} : S' \rightarrow S'$, $R(a) = b$, such that *i*) it splits S' into two hemispheres $\text{Hem}(a)$ and $\text{Hem}(b)$, $a \in \text{Hem}(a)$, $b \in \text{Hem}(b)$, in such a way that $W'(c - R(d)) < W'(c - d)$ implies c and d belong to the same hemisphere, and *ii*) R preserves the compatibility of neighbouring vertices when applied to both spins. These extensions are explored in detail in [FSL].

In the dilute Potts model with q spins, the Hamiltonian is given by $\sum_{\langle xy \rangle} J_{\langle xy \rangle} \mathbf{1}[\sigma_x \neq \sigma_y]$, and the associated random-cluster probability is defined on $\{0, 1\}^\varepsilon$; see [GHM01, Gri06], for example. In this case, if σ and ω are compatible, the values of σ_x and σ_y must coincide whenever $\omega_{\langle xy \rangle} = 1$, and there are no restrictions if $\omega_{\langle xy \rangle} = 0$. Call a connected component of the graph $(V, \{\langle xy \rangle : \omega_{\langle xy \rangle} = 1\})$ an ω -cluster. Then $\omega \preceq \sigma$ implies that σ is constant over each of the ω -clusters and the values achieved on different clusters not connected with U can take any value in $\{1, \dots, q\}$. Hence, for the diluted Potts model, the combinatorial term appearing in expression (15) reduces to

$$|\{\sigma : \sigma \succeq \omega, \sigma \stackrel{U}{=} 0\}| = q^{\text{number of } \omega\text{-clusters}}. \quad (24)$$

In contrast, for the clock model, the larger range of edge-configurations in \mathcal{I}^ε gives rise to a more delicate combinatorial structure which will be managed using the inequality (22).

3 Proofs

Proof of Theorem 1.1: Phase co-existence. Let us identify a disorder J defined in (1) with its associated set of open edges

$$\{\langle xy \rangle \in \mathcal{E}(\mathbb{Z}^d) : J_{\langle xy \rangle} = 1\}. \quad (25)$$

We say that $C \subset \mathbb{Z}^d$ is a J -open cluster if it is a maximal set with the property that $x \xleftrightarrow{J} y$ for all $x, y \in C$. Denote $x \xleftrightarrow{J} \infty$ when x belongs to an infinite J -open cluster. Let p_c be the critical value for bond percolation in \mathbb{Z}^d . If $p > p_c$ then $P_p(J : x \xleftrightarrow{J} \infty) > 0$; see [GHM01, Gri06] and references therein for a treatment of percolation theory.

Let $\rho \in (0, 1)$ be such that $p\rho > p_c$ and let J' be an independently sampled P_ρ -disorder. Denote by JJ' the set of vertices that are open for both J and J' , note that JJ' is a $P_{p\rho}$ -disorder. Also, once J is fixed, JJ' is a random thinning, each open edge of J is kept open with probability ρ and closed with probability $(1 - \rho)$, independently.

Let $\mathcal{X} \subset \{0, 1\}^{\mathcal{E}(\mathbb{Z}^d)}$ be the set of disorders J such that there is an infinite JJ' -open cluster with probability 1:

$$\mathcal{X} := \{J : P_\rho(J' : \text{there is an infinite } JJ'\text{-open cluster}) = 1\}. \quad (26)$$

From Fubini's Theorem, the fact that JJ' is a $P_{p\rho}$ -disorder, and $p\rho > p_c$, it is easy to see that $P_p(\mathcal{X}) = 1$. Also,

$$\{J' : \text{there is an infinite } JJ'\text{-open cluster}\} = \bigcup_{x \in \mathbb{Z}^d} \{J' : x \xleftrightarrow{JJ'} \infty\}.$$

Hence, for each $J \in \mathcal{X}$, there exists a vertex $x \in \mathbb{Z}^d$ belonging to an infinite JJ' -open cluster with positive P_ρ -probability:

$$P_\rho(J' : x \xleftrightarrow{JJ'} \infty) > 0. \quad (27)$$

Let $\beta_0 = \beta_0(\rho)$ be as in the statement of Theorem 2.3. Fix a disorder $J \in \mathcal{X}$ and a vertex x satisfying (27). Given $n \in \mathbb{N}$, let $\Lambda_n := [-n, n]^d \cap \mathbb{Z}^d$ and consider the choices

$$V = \Lambda_n \cup \partial\Lambda_n, \quad \mathcal{E} = \{\langle xy \rangle \in \mathcal{E}(\mathbb{Z}^d) : \{x, y\} \cap \Lambda_n \neq \emptyset, J_{\langle xy \rangle} = 1\}, \quad U = \partial\Lambda_n, \quad (28)$$

for the vertex, edge and boundary sets in Section 2. Let μ , ϕ and B_ρ denote the clock probability on S^V , random-cluster distribution on $\mathcal{I}^\mathcal{E}$ and product Bernoulli probability on $\{0, 1\}^\mathcal{E}$ associated to this choice, respectively. Note that $\mu = \mu_{\Lambda_n, J}^0$ as defined in (4) with the convention that the superscript a in $\mu_{\Lambda_n, J}^a$ indicates the boundary condition $\eta_y \equiv a$ on $\partial\Lambda_n$.

Since the event $\{x \xleftrightarrow{\omega} U\}$ is increasing, Theorem 2.3 implies

$$\phi(\omega : x \xleftrightarrow{\omega} U) \geq B_\rho(\omega : x \xleftrightarrow{\omega} U) = P_\rho(J' : x \xleftrightarrow{JJ'} \partial\Lambda_n) \geq P_\rho(J' : x \xleftrightarrow{JJ'} \infty).$$

Replacing in (21) with $\mu = \mu_{\Lambda_n, J}^0$, we obtain

$$\mu_{\Lambda_n, J}^0(\sigma : \sigma_x = 0) \geq \mu_{\Lambda_n, J}^0(\sigma : \sigma_x = a) + P_\rho(J' : x \xleftrightarrow{JJ'} \infty), \quad \text{for any } a \neq 0. \quad (29)$$

We conclude that any weak limit μ_J^0 of $\mu_{\Lambda_n, J}^0$ as $n \rightarrow \infty$ will satisfy

$$\mu_J^0(\sigma : \sigma_x = 0) > \mu_J^0(\sigma : \sigma_x = a) \quad \text{for any } a \neq 0. \quad (30)$$

By the rotational symmetry in the set S of spins, the same holds with any boundary condition b : the weak limit μ_J^b assigns maximal probability to having a spin b at x , $\mu_J^b(\sigma : \sigma_x = b) > \mu_J^b(\sigma : \sigma_x = a)$, $a \neq b$, and therefore the q -Gibbs measures μ_J^b , $b \in S$, must be different. \square

Proof of Proposition 2.2: Positive correlations. For any spin $a \neq 0$, by (19) and the fact that $x \xleftrightarrow{\omega} U$ implies $\sigma(x) = 0$,

$$\begin{aligned} \mu(\sigma : \sigma_x = a) &= \sum_{\omega : x \not\xleftrightarrow{\omega} U} \frac{\hat{\mu}(\sigma : \sigma_x = a, \omega \preceq \sigma)}{\hat{\mu}(\sigma : \omega \preceq \sigma)} \phi(\omega) \\ &\leq \sum_{\omega : x \not\xleftrightarrow{\omega} U} \frac{\hat{\mu}(\sigma : \sigma_x = 0, \omega \preceq \sigma)}{\hat{\mu}(\sigma : \omega \preceq \sigma)} \phi(\omega) = Q((\omega, \sigma) : \sigma_x = 0, x \not\xleftrightarrow{\omega} U), \end{aligned}$$

where the inequality holds by (23). Apply (20) to conclude. \square

Proof of Theorem 2.3: Stochastic domination. The measure ϕ gives positive probability to every edge configuration. Under this hypothesis, Holley's inequality (Theorem 4.8 of [GHM01] for instance), asserts that the stochastic domination $B_\rho \leq_{st} \phi$ follows from the single-bond inequalities

$$\rho \leq \phi(\omega : \omega_{\langle xy \rangle} = 1 \mid \omega : \omega \stackrel{\mathcal{E} \setminus \langle xy \rangle}{=} \omega') =: \alpha(\langle xy \rangle, \omega'), \quad \langle xy \rangle \in \mathcal{E}, \quad \omega' \in \mathcal{I}^\mathcal{E}. \quad (31)$$

Given $t \in \mathcal{I}$, we define $t_{\langle xy \rangle} \omega' \in \mathcal{I}^\mathcal{E}$ by

$$(t_{\langle xy \rangle} \omega')_{\langle xy \rangle} = t \quad \text{and} \quad t_{\langle xy \rangle} \omega' \stackrel{\mathcal{E} \setminus \langle xy \rangle}{=} \omega'.$$

Omitting the dependence of α on $(\langle xy \rangle, \omega')$ in the notation,

$$\alpha = \frac{\phi(1_{\langle xy \rangle} \omega')}{\sum_{i=0}^k \phi((t_i)_{\langle xy \rangle} \omega')} = \frac{r_k |\{\sigma : \sigma \succeq 1_{\langle xy \rangle} \omega', \sigma \stackrel{U}{=} 0\}|}{\sum_{i=0}^k r_i |\{\sigma : \sigma \succeq (t_i)_{\langle xy \rangle} \omega', \sigma \stackrel{U}{=} 0\}|}, \quad (32)$$

and

$$\alpha^{-1} = \sum_{i=0}^k \frac{r_i}{r_k} \frac{|\{\sigma : \sigma \succeq (t_i)_{\langle xy \rangle} \omega', \sigma \stackrel{U}{=} 0\}|}{|\{\sigma : \sigma \succeq 1_{\langle xy \rangle} \omega', \sigma \stackrel{U}{=} 0\}|}. \quad (33)$$

Let $(V, \tilde{\mathcal{E}})$ be the auxiliary graph obtained from (V, \mathcal{E}) by adding all edges connecting vertices in U :

$$\tilde{\mathcal{E}} := \mathcal{E} \cup \{ \langle uv \rangle : \{u, v\} \subset U \}. \quad (34)$$

Let $\tilde{\omega} \in \mathcal{I}^{\tilde{\mathcal{E}}}$ be defined by

$$\tilde{\omega} \stackrel{\tilde{\mathcal{E}} \setminus \mathcal{E}}{=} 1 \quad \text{and} \quad \tilde{\omega} \stackrel{\mathcal{E}}{=} \omega'.$$

Extend the definition of $t_{\langle xy \rangle} \tilde{\omega} \in \mathcal{I}^{\tilde{\mathcal{E}}}$ and the compatibility notion $\sigma \preceq \tilde{\omega}$ to the enlarged graph in the obvious way and use the rotation invariance of S to get

$$|\{\sigma : \sigma \succeq (t_i)_{\langle xy \rangle} \omega', \sigma \stackrel{U}{=} 0\}| = \frac{1}{q} |\{\sigma : \sigma \succeq (t_i)_{\langle xy \rangle} \tilde{\omega}\}|, \quad (35)$$

and replacing in (33),

$$\alpha^{-1} = \sum_{i=0}^k \frac{r_i}{r_k} \frac{|\{\sigma : \sigma \succeq (t_i)_{\langle xy \rangle} \tilde{\omega}\}|}{|\{\sigma : \sigma \succeq 1_{\langle xy \rangle} \tilde{\omega}\}|}. \quad (36)$$

For $0 \leq i \leq k$, let

$$K_i := |\{(a, b) \in S \times S : W(a - b) = t_i\}|. \quad (37)$$

We have

$$|\{\sigma : \sigma \succeq (t_i)_{\langle xy \rangle} \tilde{\omega}\}| = \sum_{j=i}^k |\{\sigma : \sigma \succeq \tilde{\omega}, W(\sigma_y - \sigma_x) = t_j\}| \quad (38)$$

$$= \sum_{j=i}^k K_j |\{\sigma : \sigma \succeq \tilde{\omega}, \sigma_y = 0, \sigma_x = a_j\}|, \quad (39)$$

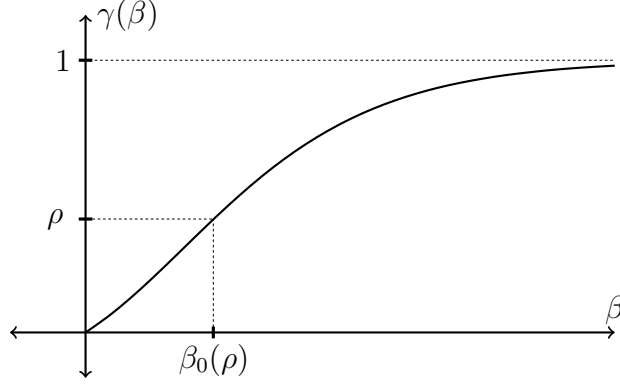


Figure 1

where $a_j \in S$ is an angle such that $W(a_j) = t_j$. The second identity holds again by rotation invariance. Replacing in expression (36),

$$\alpha^{-1} = \sum_{i=0}^k \sum_{j=i}^k \frac{r_i}{r_k} \frac{K_j}{K_k} \frac{|\{\sigma : \sigma \succeq \tilde{\omega}, \sigma_y = 0, \sigma_x = a_j\}|}{|\{\sigma : \sigma \succeq \tilde{\omega}, \sigma_y = 0, \sigma_x = 0\}|}. \quad (40)$$

By Lemma 2.4 applied to $U = \{y\}$ we get

$$\alpha^{-1} \leq \sum_{i=0}^k \sum_{j=i}^k \frac{r_i}{r_k} \frac{K_j}{K_k} = \sum_{j=0}^k \frac{t_j}{r_k} \frac{K_j}{K_k}, \quad (41)$$

since $t_j = \sum_{i=0}^j r_i$. From (31), we conclude that the stochastic domination $B_\rho \leq_{st} \phi$ will follow for β satisfying

$$\rho \leq \gamma(\beta) := \left(\sum_{j=0}^k \frac{t_j}{r_k} \frac{K_j}{K_k} \right)^{-1}. \quad (42)$$

The function γ is increasing. Indeed, for each j , $\frac{r_k}{t_j}$ is of the form $e^{\beta A}(1 - e^{-\beta B})$ with A and B positive numbers, and hence increasing. On the other hand $\lim_{\beta \rightarrow \infty} r_k = 1$ and $\lim_{\beta \rightarrow \infty} t_i = 0$ for $i < k$; as a consequence, $\lim_{\beta \rightarrow \infty} \gamma(\beta) = 1$. Finally, $\lim_{\beta \downarrow 0} r_k = 0$ and $\lim_{\beta \downarrow 0} t_i = 1$ for every i , so $\lim_{\beta \downarrow 0} \gamma(\beta) = 0$. See Figure 1 for the graph of γ when $q = 4$. In particular, γ is injective and its inverse $\gamma^{-1} : (0, 1) \rightarrow (0, \infty)$ is well defined. We conclude that if $\beta_0 = \gamma^{-1}(\rho)$, then equation (31) holds for $\beta \geq \beta_0$. \square

Proof of Lemma 2.4. The case $x \in U$ is trivial, so let us suppose $x \in V \setminus U$. If $|U| > 1$ the model can be reduced to the case $|U| = 1$ by identifying all vertices in U . We may then suppose $U = \{y\}$ for some $y \neq x$.

Let

$$L_\omega(a) := \{\sigma : \sigma \succeq \omega, \sigma_y = 0, \sigma_x = a\}. \quad (43)$$

We will construct an injection $F : L_\omega(a) \hookrightarrow L_\omega(0)$. Here is a brief description of the procedure. Fix $a \in S$ and consider the reflection $R : S \rightarrow S$ with respect to the line ℓ at angle $a/2$ with the horizontal axis (see Figure 2), that is, $Rb = a - b \bmod 2\pi$. Clearly, $R(a) = 0$. We progressively transform an initial configuration $\sigma \in L_\omega(a)$ into a configuration $\sigma' \in L_\omega(0)$. The first step is to modify σ by applying the reflection R to the spin at the vertex x . The resulting

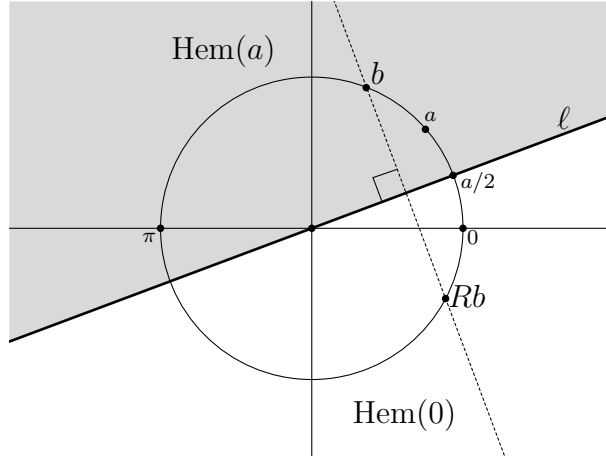


Figure 2

configuration may present incompatibilities with respect to ω and, if it does, they will appear at edges $\{\langle ux \rangle\}_{u \in V}$. If this is the case, we modify the configuration by applying the transformation R to the spins of the conflicting vertices. We obtain a configuration without incompatibilities in the edges having one endpoint at x , but we might have created new incompatibilities at a second level of edges, that is, edges with one endpoint at a vertex that is a neighbour of x . We solve this by applying R once more to the spins of the new conflicting vertices, and keep repeating the procedure until there are no more incompatibilities. We need to show that the resulting configuration σ' belongs to $L_\omega(0)$, and that the construction is indeed injective. The most delicate part is to prove that this process stops before reaching the vertex y .

It suffices to prove the result when $a \neq 0$, which we assume from now on. We may also assume that $a \in (0, \pi]$, as the other case is symmetric. As before, the boundary $\partial V'$ of a vertex set $V' \subset V$ denotes the set of vertices $u \in V \setminus V'$ such that $\langle uv \rangle \in \mathcal{E}$ for some $v \in V'$.

Let now $\sigma \in L_\omega(a)$. Define a sequence of sets $A_0 \subset A_1 \subset \dots \subset V$ associated to σ by $A_0 := \{x\}$ and, for $n \geq 0$,

$$A_{n+1} := A_n \cup \{u \in \partial A_n : W(\sigma_u - R\sigma_v) < \omega_{\langle uv \rangle} \text{ for some } v \in A_n\}. \quad (44)$$

At each step, $A_{n+1} \setminus A_n$ consists of those vertices where new incompatibilities would arise when applying the reflection to A_n . Let

$$A := \bigcup_{n \geq 0} A_n. \quad (45)$$

Define the function $F : L_\omega(a) \rightarrow S^V$ by

$$(F\sigma)_u := \begin{cases} R\sigma_u & \text{if } u \in A \\ \sigma_u & \text{if } u \notin A. \end{cases} \quad (46)$$

We now show that *i)* the image of F is contained in $L_\omega(0)$ and *ii)* that $F : L_\omega(a) \rightarrow L_\omega(0)$ is an injection.

i) $F\sigma \in L_\omega(0)$. In order to prove that $F\sigma \succeq \omega$, we need to show that

$$W((F\sigma)_u - (F\sigma)_v) \geq \omega_{\langle uv \rangle} \quad (47)$$

for any $\langle uv \rangle \in \mathcal{E}$. The cases $\{u, v\} \subset A$ or $\{u, v\} \subset A^c$ are trivial. If $u \notin A$ and $v \in A$, condition (47) reads $W(\sigma_u - R\sigma_v) \geq \omega_{\langle uv \rangle}$, which must hold; otherwise u would have belonged to A in the first place.

It remains to prove that $(F\sigma)_y = 0$, which follows if we show that $y \notin A$. The line ℓ (see Figure 2) separates the two open hemispheres $\text{Hem}(0)$ and $\text{Hem}(a)$ defined by

$$\begin{aligned}\text{Hem}(0) &:= \{b \in S : \sin(b - a/2) < 0\} \\ \text{Hem}(a) &:= \{b \in S : \sin(b - a/2) > 0\}.\end{aligned}$$

Since $0 \in \text{Hem}(0)$, it is enough to prove that $\sigma_u \in \text{Hem}(a)$ for every $u \in A \setminus \{x\}$. We proceed by induction. If $A_1 \neq \emptyset$, let $u \in A_1 \setminus \{x\}$ with $\sigma_u = b$. By the definition of A_1 , we have $W(b - 0) < \omega_{\langle ux \rangle} \leq W(b - a)$, where the inequality follows from the fact that $\sigma \succeq \omega$. Now, $W(b - 0) < W(b - a)$ is equivalent to $\cos(b) < \cos(b - a)$. But

$$\cos(b) < \cos(b - a) \iff \cos\left(b - \frac{a}{2} + \frac{a}{2}\right) < \cos\left(b - \frac{a}{2} - \frac{a}{2}\right) \quad (48)$$

$$\begin{aligned}\iff \cos\left(b - \frac{a}{2}\right) \cos\left(\frac{a}{2}\right) - \sin\left(b - \frac{a}{2}\right) \sin\left(\frac{a}{2}\right) \\ < \cos\left(b - \frac{a}{2}\right) \cos\left(\frac{a}{2}\right) + \sin\left(b - \frac{a}{2}\right) \sin\left(\frac{a}{2}\right)\end{aligned} \quad (49)$$

$$\iff 0 < 2 \sin\left(b - \frac{a}{2}\right) \sin\left(\frac{a}{2}\right) \iff 0 < \sin\left(b - \frac{a}{2}\right) \iff b \in \text{Hem}(a), \quad (50)$$

and the claim holds for A_1 . Suppose now that $\sigma_u \in \text{Hem}(a)$, that is

$$\sin\left(\sigma_u - \frac{a}{2}\right) > 0, \quad (51)$$

for every $u \in A_n$. If $A_{n+1} \neq \emptyset$, let $v \in A_{n+1}$ and $w \in A_n$ be such that $W(\sigma_v - R\sigma_w) < W(\sigma_v - \sigma_w)$, which is equivalent to $\cos(\sigma_v - (a - \sigma_w)) < \cos(\sigma_v - \sigma_w)$. By the inductive hypothesis $\sin(\sigma_w - \frac{a}{2}) > 0$. An argument similar to the one leading from (48) to (50) yields

$$0 < 2 \sin\left(\sigma_v - \frac{a}{2}\right) \sin\left(\sigma_w - \frac{a}{2}\right),$$

and then $0 < \sin(\sigma_v - \frac{a}{2})$, i.e. $\sigma_v \in \text{Hem}(a)$. This completes the induction.

ii) F is injective. Let $\sigma, \sigma' \in L_\omega(0, a)$ be two different configurations and denote by A, A_1, A_2, \dots and A', A'_1, A'_2, \dots their associated incompatibility sets. If $A = A'$, we are done because R is injective. Suppose $A \neq A'$ and let

$$n = \min \{j \geq 1 : A_j \neq A'_j\};$$

so that in particular $A_{n-1} = A'_{n-1}$. If there is a vertex $u \in A_{n-1}$ such that $\sigma_u \neq \sigma'_u$, we are done. Suppose $\sigma \stackrel{A_{n-1}}{=} \sigma'$. Without loss of generality, let us take $u \in A_n \setminus A'_n$. We claim that $(F\sigma)_u \neq (F\sigma')_u$. We know that $\sigma_u \neq \sigma'_u$, as otherwise we would have $u \in A'_n$. If $u \in A'$ we are done. Suppose then $u \notin A'$. Let $v \in A_{n-1}$ be such that $W(\sigma_u - R\sigma_v) < \omega_{\langle uv \rangle}$. Using that $W(a' - b') = W(Ra' - Rb')$ for any $a', b' \in S$, that R^2 is the identity and that $(F\sigma)_u = R\sigma_u$, we have $W(\sigma_u - R\sigma_v) = W((F\sigma)_u - \sigma_v)$, and then

$$W((F\sigma)_u - \sigma_v) < \omega_{\langle uv \rangle}. \quad (52)$$

On the other hand, since $\sigma' \succeq \omega$, we have $W(\sigma'_u - \sigma'_v) \geq \omega_{\langle uv \rangle}$. But $W(\sigma'_u - \sigma'_v) = W((F\sigma')_u - \sigma_v)$ because $u \notin A'$ and $\sigma \stackrel{A_{n-1}}{=} \sigma'$, and hence

$$W((F\sigma')_u - \sigma_v) \geq \omega_{\langle uv \rangle}. \quad (53)$$

From inequalities (52) and (53) we obtain $(F\sigma)_u \neq (F\sigma')_u$, as claimed. \square

4 Appendix

Proof of Theorem 2.1: The Edwards-Sokal random-cluster representation. The first step is to write the density of μ with respect to $\hat{\mu}$:

$$\mu(\sigma) = \hat{\mu}(\sigma) \frac{1}{Z'} \prod_{\langle xy \rangle \in \mathcal{E}} W(\sigma_x - \sigma_y), \quad (54)$$

with Z' the normalizing constant in (17). Since $W(\sigma_x - \sigma_y) = \theta(t \in \mathcal{I} : t \leq W(\sigma_x - \sigma_y))$, the weight of a spin configuration can be realized as the probability of a related event on the associated edge set:

$$\prod_{\langle xy \rangle \in \mathcal{E}} W(\sigma_x - \sigma_y) = \hat{\phi} \left(\bigcap_{\langle xy \rangle \in \mathcal{E}} \{ \omega \in \mathcal{I}^{\mathcal{E}} : \omega_{\langle xy \rangle} \leq W(\sigma_x - \sigma_y) \} \right) = \hat{\phi}(\omega : \omega \preceq \sigma).$$

Here is where the definition of compatibility appears naturally. Inserting (55) in (54), we get

$$\mu(\sigma) = \sum_{\omega \in \mathcal{I}^{\mathcal{E}}} \frac{1}{Z'} \mathbf{1}[\omega \preceq \sigma] \hat{\phi}(\omega) \hat{\mu}(\sigma) = \sum_{\omega \in \mathcal{I}^{\mathcal{E}}} Q((\omega, \sigma)). \quad (55)$$

Hence, μ is the second marginal of Q . Adding over all the possible vertex-configurations, it is easy to see that ϕ is its first marginal. \square

Asymptotics for β_0 . The threshold β_0 introduced in Theorem 1.1 is $\beta_0 = \gamma^{-1}(\rho)$, where $\gamma : (0, \infty) \rightarrow (0, 1)$ is the function defined in the proof of Theorem 2.3, and ρ is the parameter defined in the proof of Theorem 1.1, such that $\rho > \frac{p_c}{p}$. Since γ^{-1} is increasing, we can take the infimum over ρ to optimize $\beta_0 = \gamma^{-1}(\frac{p_c}{p})$.

For any fixed $\beta > 0$ we have that $\lim_{q \rightarrow \infty} \gamma(\beta) = 0$. Indeed, $\lim_{q \rightarrow \infty} r_k = 0$ and $t_i \frac{K_i}{K_k}$ is bounded away from zero uniformly in q . As a consequence, for every fixed $\tilde{p} \in (0, 1)$, $\lim_{q \rightarrow \infty} \gamma^{-1}(\tilde{p}) = \infty$. We conclude that our method is not informative as a discretization of the XY model, that is, when the number of spins goes to infinity.

Note that

$$\gamma(\beta)^{-1} = \frac{1}{r_k} + \sum_{i=0}^{k-1} \frac{t_i}{r_k} \frac{K_i}{K_k} \leq \frac{1}{r_k} + \sum_{i=0}^{k-1} \frac{t_{k-1}}{r_k} 2 \leq \frac{1}{r_k} + \frac{t_{k-1}}{r_k} q, \quad (56)$$

so that

$$\gamma(\beta) \geq \frac{r_k}{1 + q t_{k-1}}. \quad (57)$$

Then β_0 is bounded above by the solution to the equation

$$\frac{p_c}{p} = \frac{r_k}{1 + q t_{k-1}}. \quad (58)$$

Using that $r_k = 1 - t_{k-1}$ and that $t_{k-1} = e^{-\beta(1 - \cos(\frac{2\pi}{q}))}$, this solution can be explicitly computed as

$$\frac{\log \left(\frac{p + q p_c}{p - p_c} \right)}{1 - \cos \left(\frac{2\pi}{q} \right)}. \quad (59)$$

If we fix p and d , this expression is of order $q^2 \log(q)$ as $q \rightarrow \infty$, the same order given by Pirogov-Sinai theory and reflection positivity in the 2-dimensional homogeneous case. If we fix p and q , it is of order

$$\frac{\log\left(1 + \frac{1}{d}\right)}{1 - \cos\left(\frac{2\pi}{q}\right)} \quad (60)$$

as $d \rightarrow \infty$, taking into account that $p_c \sim \frac{1}{2d}$. In particular, $\beta_0 \rightarrow 0$ as $d \rightarrow \infty$.

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